

## Shallow Water Quasi-Geostrophic Theory on the Sphere

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Quasi-geostrophic theory forms the basis for much of our understanding of mid-latitude atmospheric dynamics. The theory is typically presented in either its  $f$ -plane form or its  $\beta$ -plane form. However, for many applications, including diagnostic use in global climate modeling, a fully spherical version would be most useful. Such a global theory does in fact exist and has for many years, but few in the scientific community seem to have ever been aware of it. In the context of shallow water dynamics, it is shown that the spherical version of quasi-geostrophic theory is easily derived (re-derived) based on a partitioning of the flow between nondivergent and irrotational components, as opposed to a partitioning between geostrophic and ageostrophic components. In this way, the invertibility principle is expressed as a relation between the streamfunction and the potential vorticity, rather than between the geopotential and the potential vorticity. This global theory is then extended by showing that the invertibility principle can be solved analytically using spheroidal harmonic transforms, an advancement that greatly improves the usefulness of this “forgotten” theory. When the governing equation for the time evolution of the potential vorticity is linearized about a state of rest, a simple Rossby-Haurwitz wave dispersion relation is derived and examined. These waves have a horizontal structure described by spheroidal harmonics, and the Rossby-Haurwitz wave frequencies are given in terms of the eigenvalues of the spheroidal harmonic operator. Except for sectoral harmonics with low zonal wavenumber, the quasi-geostrophic Rossby-Haurwitz frequencies agree very well with those calculated from the primitive equations. One of the many possible applications of spherical quasi-geostrophic theory is to the study of quasi-geostrophic turbulence on the sphere. In this context, the theory is used to derive an anisotropic Rhines barrier in three-dimensional wavenumber space.

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### 1. Introduction

One of the barriers to progress in tropical dynamics is the underdeveloped state of potential vorticity (PV) arguments and interpretations of balanced tropical flows. While it is true that certain aspects of tropical flows, such as the inertia-gravity wave and Kelvin wave contributions, are not controlled by the evolving PV field, it is also true that a major part of many tropical flows is quasi-geostrophic in character and under control of the PV field (Schubert and Masarik 2006). Quasi-geostrophic theory is typically developed and applied in either its  $f$ -plane form or its  $\beta$ -plane form. A fully spherical version would help provide a unified view of midlatitude and tropical disturbances and would be very useful for a number of applications, including diagnostic use in global climate modeling. The existence of such a spherical version, however, was unknown to most in the scientific community (including the authors of this paper). It wasn't until after we had successfully derived a fully spherical version of quasi-geostrophic theory, that we discovered that this theory was not actually new, but in fact had been formulated long ago by Kuo (1959, see his equations 5 and 6) and (more clearly) by Charney and Stern (1962, see their equation 2.25). Both of these groups actually derived the spherical version for a fully stratified atmosphere,

but these results easily lead to the same shallow water version as is presented here. For some reason these early results were not well known or were forgotten. It is interesting that even Charney and Stern (1962) did not reference Kuo (1959) even though both groups had ties to the same institution.

The purpose of this paper is to re-introduce the scientific community to a fully spherical version of quasi-geostrophic theory and then to advance the usefulness of this theory through the application of spheroidal harmonics, to test this theory by comparing Rossby-Haurwitz wave dispersion results with those from the primitive equations, and to apply it to the study of turbulence on the sphere. In Section 2 we present the shallow water version of this theory by showing our derivation of it, a derivation that involves the direct approximation of the exact PV equation. This approach is more direct than that taken by Kuo and by Charney and Stern, but leads to the same fundamental result. Although the resulting PV conservation relation is nonlinear, the associated invertibility principle is linear and involves an elliptic “spheroidal

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operator." This latter observation allows us to advance this theory as discussed in Section 3, for it enables us to solve the invertibility principle analytically using spheroidal harmonics. One way to assess the accuracy of the global quasi-geostrophic theory is to derive its Rossby-Haurwitz wave frequencies and then compare them with the "exact" frequencies obtained numerically from the primitive equations. Such a test of the theory is performed in Section 4. There are many applications in which global quasi-geostrophic theory can provide new insights. The study of quasi-geostrophic turbulence and the emergence of zonal jets is one such application as presented in Section 5. For this problem, global quasi-geostrophic theory provides a simple framework to generalize the concept of the anisotropic Rhines barrier. Finally, some historical remarks are given in Section 6.

## 2. Shallow water quasi-geostrophic theory on the sphere

In order to provide a simple and straightforward derivation of spherical quasi-geostrophic theory, consider motions of a shallow water fluid on the sphere. Using spherical coordinates  $(\lambda, \phi)$  we can write the shallow water primitive equations as

$$\frac{Du}{Dt} - \left(2\Omega \sin \phi + \frac{u \tan \phi}{a}\right) v + g \frac{\partial h}{a \cos \phi \partial \lambda} = 0, \quad (2.1)$$

$$\frac{Dv}{Dt} + \left(2\Omega \sin \phi + \frac{u \tan \phi}{a}\right) u + g \frac{\partial h}{a \partial \phi} = 0, \quad (2.2)$$

$$\frac{Dh}{Dt} + (\bar{h} + h) \left( \frac{\partial u}{a \cos \phi \partial \lambda} + \frac{\partial(v \cos \phi)}{a \cos \phi \partial \phi} \right) = 0, \quad (2.3)$$

where  $u$  is the eastward component of velocity,  $v$  the northward component,  $h$  the deviation of the fluid depth from the global mean depth  $\bar{h}$ ,  $\Omega$  the earth's rotation rate,  $a$  the earth's radius,  $g$  the acceleration of gravity, and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{a \cos \phi \partial \lambda} + v \frac{\partial}{a \partial \phi} \quad (2.4)$$

the material derivative. The potential vorticity equation, derived from (2.1)–(2.3), is

$$\frac{DP}{Dt} = 0, \quad (2.5)$$

where  $P$  is the potential vorticity given by

$$\begin{aligned} P &= \frac{\bar{h}}{\bar{h} + h} \left( 2\Omega \sin \phi + \frac{\partial v}{a \cos \phi \partial \lambda} - \frac{\partial(u \cos \phi)}{a \cos \phi \partial \phi} \right) \\ &= 2\Omega \mu + \left( \frac{\bar{h}}{\bar{h} + h} \right) \nabla^2 \psi - 2\Omega \mu \left( \frac{h}{\bar{h} + h} \right), \end{aligned} \quad (2.6)$$

with  $\mu = \sin \phi$  and  $\psi$  denoting the streamfunction for the non-divergent part of the flow

$$\begin{aligned} (u_\psi, v_\psi) &= \left( -\frac{\partial \psi}{a \partial \phi}, \frac{\partial \psi}{a \cos \phi \partial \lambda} \right) \\ &= \left( -(1 - \mu^2)^{1/2} \frac{\partial \psi}{a \partial \mu}, (1 - \mu^2)^{-1/2} \frac{\partial \psi}{a \partial \lambda} \right). \end{aligned} \quad (2.7)$$

The derivations given in Kuo (1959) and Charney and Stern (1962) involve approximation of the vorticity and thermodynamic equations. Here we follow a more direct route via the potential vorticity equation (2.5). We first assume  $|h| \ll \bar{h}$  and then approximate (2.6) by

$$P \approx 2\Omega \mu + \nabla^2 \psi - \frac{2\Omega \mu}{\bar{h}} h = 2\Omega \mu + q, \quad (2.8)$$

where  $q$  is the potential vorticity anomaly. Next, we need to formulate a balance condition between the mass field  $h$  and the nondivergent wind field  $\psi$ . Such a balance condition will convert (2.8) into an invertibility principle, i.e., a relation between  $q$  and either  $h$  or  $\psi$ . Following the arguments of Kuo (1959) and Charney and Stern (1962), we now assume (i) that  $\psi$  and  $h$  are related by the linear balance condition

$$\nabla \cdot (2\Omega \mu \nabla \psi) = g \nabla^2 h, \quad (2.9)$$

and (ii) that  $2\Omega \mu$  can be considered as slowly varying, so that (2.9) can be simplified to

$$\nabla^2 (gh - 2\Omega \mu \psi) = 0, \quad (2.10)$$

from which the local linear balance condition

$$gh = 2\Omega \mu \psi \quad (2.11)$$

then follows. Using (2.11) and approximating the advecting velocity in (2.4) by  $(u_\psi, v_\psi)$ , substitution of (2.8) into (2.5) yields

$$\frac{\partial q}{\partial t} + \frac{1}{a^2} \frac{\partial(\psi, q)}{\partial(\lambda, \mu)} + \frac{2\Omega}{a^2} \frac{\partial \psi}{\partial \lambda} = 0, \quad (2.12)$$

where

$$\nabla^2 \psi - \frac{\varepsilon \mu^2}{a^2} \psi = q \quad (2.13)$$

is now the quasi-geostrophic potential vorticity anomaly, and where

$$\varepsilon = \frac{4\Omega^2 a^2}{g\bar{h}} = \left( \frac{a}{(g\bar{h})^{1/2}/(2\Omega)} \right)^2 \quad (2.14)$$

is Lamb's parameter, which, according to the second equality, can be interpreted as the square of the ratio of the earth's radius to the Rossby radius of deformation. Equations (2.12) and (2.13) form a closed system in  $q$  and  $\psi$ , and they constitute the shallow water version of global quasi-geostrophic theory. Note that the term "quasi-geostrophic theory" is used here in a general sense, since the relation between the mass field and the nondivergent wind field given by (2.11) is not strictly equivalent to geostrophic balance. The term "local linear balance theory" might also be appropriate. The system (2.12)–(2.13) possesses a reasonable total energy principle, which is discussed in Section 5.

### 3. Spheroidal harmonics

The PV conservation relation (2.12) is nonlinear, but the invertibility principle (2.13) is linear and can be solved analytically using spheroidal harmonics. The introduction of spheroidal harmonics will allow for straightforward analyses of the dispersion properties of Rossby-Haurwitz waves (Section 4) and the cascades of energy and potential enstrophy (Section 5). The spheroidal harmonics  $\mathcal{S}_{mn}(\varepsilon; \lambda, \mu)$  satisfy

$$\nabla^2 \mathcal{S}_{mn} - \frac{\varepsilon \mu^2}{a^2} \mathcal{S}_{mn} = -\frac{\alpha_{mn}(\varepsilon)}{a^2} \mathcal{S}_{mn}, \quad (3.1)$$

where  $m$  is the zonal wavenumber,  $n$  is the total wavenumber, and  $-\alpha_{mn}(\varepsilon)/a^2$  is the eigenvalue of the operator  $\nabla^2 - \varepsilon \mu^2/a^2$ . Spheroidal harmonics have the separable form

$$\mathcal{S}_{mn}(\varepsilon; \lambda, \mu) = S_{mn}(\varepsilon; \mu) e^{im\lambda}, \quad (3.2)$$

which, when substituted in (3.1), yields the ordinary differential equation

$$\frac{d}{d\mu} \left[ (1 - \mu^2) \frac{dS_{mn}}{d\mu} \right] + \left( \alpha_{mn} - \varepsilon \mu^2 - \frac{m^2}{1 - \mu^2} \right) S_{mn} = 0 \quad (3.3)$$

for the meridional structure functions  $S_{mn}(\varepsilon; \mu)$ . Equation (3.3) is known as the ‘‘spheroidal wave equation,’’ and its solutions (eigenfunctions) are typically referred to as ‘‘spheroidal wave functions.’’ The values of  $\alpha_{mn}(\varepsilon)$  are the eigenvalues of the spheroidal wave equation. A concise summary of spheroidal harmonics is given in Abramowitz and Stegun (1965, 751–769). More extensive discussions are given in Morse and Feshbach (1953), Stratton et al. (1956), Flammer (1957), and Meixner et al. (1980). The book by Flammer is particularly useful.

In a continuously stratified model the shallow water mean fluid depth  $\bar{h}$  is replaced by a set of equivalent depths, and a wide range of Lamb’s parameters is then possible (including negative  $\varepsilon$  for forced problems). As shown in Table 1, values of  $\varepsilon$  ranging between 10 and 10,000 lead to meaningful equivalent depths (Fulton and Schubert 1985). Unfortunately, for general  $\varepsilon$ , closed form expressions for  $\alpha_{mn}(\varepsilon)$  and  $S_{mn}(\varepsilon; \mu)$  are not available. However, very accurate approximations of  $\alpha_{mn}(\varepsilon)$  and  $S_{mn}(\varepsilon; \mu)$  can be obtained by the procedure described in Appendix A (based on Hodge 1970). We have used this procedure to produce Tables 2–5, which display values of  $\alpha_{mn}(\varepsilon)$  for  $\varepsilon = 10, 100, 1000, 10000$ . These tables are in excellent agreement with the abbreviated ( $m \leq 2$ ) tables given by Abramowitz and Stegun (1965, 760–764). Version 6 of the Mathematica software package now has built-in support for computing spheroidal harmonic eigenvalues and eigenfunctions, a feature that was also used to verify the results given in Tables 2–5.

Although closed form expressions for  $\alpha_{mn}(\varepsilon)$  and  $S_{mn}(\varepsilon; \mu)$  are not available for general values of  $\varepsilon$ , there are certain special cases where closed form expressions or asymptotic approximations are available. For example, when  $\varepsilon = 0$ ,

**Table 1.** Equivalent depths ( $\bar{h}$ ) and gravity wave speeds ( $(g\bar{h})^{1/2}$ ) associated with select values of Lamb’s parameter ( $\varepsilon$ ). Note that  $\varepsilon = 10$  roughly corresponds to the external mode, and  $\varepsilon = 10,000$  roughly corresponds to the fifth or sixth internal mode.

Lamb’s Parameter	Equivalent Depth (m)	Gravity Wave Speed (m s <sup>-1</sup> )
10	8,809.	293.8
100	880.9	92.91
1,000	88.09	29.38
10,000	8.809	9.291

the spheroidal harmonic eigenvalues become independent of  $m$ . In that case,  $\alpha_{mn}(0) = n(n+1)$  and the spheroidal harmonics reduce to the spherical harmonics, i.e.,

$$S_{mn}(0; \lambda, \mu) \sim P_n^m(\mu) e^{im\lambda}, \quad (3.4)$$

where

$$P_n^m(\mu) = (1 - \mu^2)^{m/2} \frac{d^m P_n(\mu)}{d\mu^m} \quad (3.5)$$

is the associated Legendre function, with  $P_n(\mu)$  denoting the ordinary Legendre polynomial. In this special case ( $\varepsilon = 0$ ), we see from (2.13) that there is no distinction between vorticity and potential vorticity, so that the shallow water, quasi-geostrophic model reduces to the nondivergent barotropic model. Approximate formulas for  $\alpha_{mn}(\varepsilon)$  are also available, including a power series formula valid for small values of  $\varepsilon$  and an asymptotic formula valid for large values of  $\varepsilon$ . The latter formula is of most interest here since we shall be dealing with values of  $\varepsilon$  that are greater than or equal to 10. The first few terms of the asymptotic formula yield

$$\alpha_{mn}(\varepsilon) = \varepsilon^{1/2} [1 + 2(n - m)] + m^2 - \frac{1}{8} \{ [1 + 2(n - m)]^2 + 5 \} + O(\varepsilon^{-1/2}). \quad (3.6)$$

The extension of this asymptotic formula, accurate to  $O(\varepsilon^{-3})$ , is given by Flammer (1957, p. 60) and Abramowitz and Stegun (1965, p. 754). The derivation of the first term in the asymptotic expansion (3.6) demonstrates the connection to equatorial  $\beta$ -plane theory and is given in Appendix B.

Even when  $\varepsilon \neq 0$  the spherical harmonic eigenvalues and eigenfunctions are relevant. This can be seen by noting that in (3.1) the order of magnitude of  $\nabla^2$  is  $n^2/a^2$ , while the order of magnitude of  $\varepsilon \mu^2/a^2$  is  $\varepsilon/a^2$ . Thus, for the second term in (3.1) to be small compared with the first, we must have  $n \gg \varepsilon^{1/2}$ , in which case the spheroidal harmonics are closely approximated by the spherical harmonics. For  $\varepsilon = 10, 100, 1000, 10000$  this condition is  $n \gg 3, 10, 32, 100$ . Note that in Table 2 the upper row of eigenvalues  $\alpha_{mn}(\varepsilon)$  for  $n = 10$  and  $\varepsilon = 10$  are close to the spherical harmonic value of  $n(n+1) = 110$ . For  $\varepsilon = 100, 1000, 10000$ , values of  $n$  larger than those displayed in Tables 3–5 are required in order for the spheroidal harmonic eigenvalues to approach the spherical harmonic value of  $n(n+1)$ . However, for a model with a

**Table 2.** The upper table shows eigenvalues  $\alpha_{mn}(\varepsilon)$  of the spheroidal wave equation (3.3) for  $\varepsilon = 10$ . The lower table shows quasi-geostrophic Rossby-Haurwitz wave frequencies  $\nu_{mn}(\varepsilon)/(2\Omega)$  for  $\varepsilon = 10$ . For comparison, the values in parentheses in the lower table are the primitive equation values of Longuet-Higgins (1968). The \* signifies that we have corrected an error in Longuet-Higgins' Table 5.

Eigenvalues: $\alpha_{mn}(10)$						
$n$	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
10	115.040	114.993	114.851	114.615	114.285	113.863
9	95.0497	94.9911	94.8157	94.5245	94.1190	93.6015
8	77.0625	76.9882	76.7661	76.3982	75.8877	75.2389
7	61.0810	60.9835	60.6925	60.2129	59.5518	58.7180
6	47.1096	46.9750	46.5761	45.9253	45.0402	43.9398
5	35.1576	34.9571	34.3740	33.4474	32.2210	30.7354
4	25.2513	24.9063	23.9791	22.5961	20.8565	
3	17.4674	16.7063	15.1221	13.0231		
2	11.7904	9.91621	7.26504			
1	7.28525	3.64390				
0	2.30504					

Dimensionless Frequencies: $\nu_{mn}(10)/(2\Omega)$						
$n$	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
10	0.0	0.00869618	0.0174139	0.0261747	0.0350003	0.0439125 (0.043894)
9	0.0	0.0105273	0.0210936	0.0317378	0.0424994 (0.042468)	0.0534180 (0.053390)
8	0.0	0.0129890	0.0260532	0.0392679 (0.039216)	0.0527094 (0.052658)	0.0664549 (0.066419)
7	0.0	0.0163979	0.0329530 (0.032874)	0.0498232 (0.049728)	0.0671685 (0.067086)	0.0851528 (0.085126)*
6	0.0	0.0212879 (0.021192)	0.0429405 (0.042775)	0.0653234 (0.065141)	0.0888097 (0.088704)	0.113792 (0.11391)
5	0.0	0.0286065 (0.028377)	0.0581835 (0.057808)	0.0896930 (0.089365)	0.124143 (0.12424)	0.162679 (0.16391)
4	0.0	0.0401505 (0.039534)	0.0834061 (0.082513)	0.132766 (0.13253)	0.191787 (0.19475)	
3	0.0	0.0598577 (0.058026)	0.132257 (0.13042)	0.230360 (0.23883)		
2	0.0	0.100845 (0.094951)	0.275291 (0.30561)			
1	0.0	0.274432 (0.41399)				

truncation near  $n = 200$ , all the spheroidal harmonic eigenvalues and eigenfunctions near this truncation limit will be close to their corresponding spherical harmonic values.

The orthonormality relation for spheroidal harmonics is

$$\frac{1}{4\pi} \int_{-1}^1 \int_0^{2\pi} \mathcal{S}_{mn}(\varepsilon; \lambda, \mu) \mathcal{S}_{m'n'}^*(\varepsilon; \lambda, \mu) d\lambda d\mu = \begin{cases} 1 & (m', n') = (m, n) \\ 0 & (m', n') \neq (m, n), \end{cases} \quad (3.7)$$

or, in terms of  $S_{mn}(\varepsilon; \mu)$ ,

$$\frac{1}{2} \int_{-1}^1 S_{mn}(\varepsilon; \mu) S_{m'n'}(\varepsilon; \mu) d\mu = \begin{cases} 1 & n' = n \\ 0 & n' \neq n. \end{cases} \quad (3.8)$$

For a specified value of  $\varepsilon$ , the spheroidal harmonic transform pair for the potential vorticity is

$$q_{mn}(t) = \frac{1}{4\pi} \int_{-1}^1 \int_0^{2\pi} q(\lambda, \mu, t) \mathcal{S}_{mn}^*(\varepsilon; \lambda, \mu) d\lambda d\mu, \quad (3.9)$$

**Table 3.** As in Table 2, but for  $\varepsilon = 100$ .

Eigenvalues:  $\alpha_{mn}(100)$

$n$	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
10	163.097	162.367	160.273	157.003	152.750	147.678
9	143.872	142.812	139.957	135.722	130.425	124.294
8	127.051	125.271	121.210	115.664	109.078	101.712
7	112.881	109.450	103.598	96.4081	88.3308	79.6032
6	101.035	94.6220	86.5075	77.4676	67.7973	57.6507
5	89.7393	79.7729	69.3031	58.4013	47.1437	35.5881
4	76.9933	64.1064	51.5248	38.8899	26.1251	
3	62.2577	47.3015	32.9382	18.7460		
2	45.8690	29.3389	13.4631			
1	28.1335	10.2878				
0	9.22830					

Dimensionless Frequencies:  $v_{mn}(100)/(2\Omega)$

$n$	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
10	0.0	0.00615889	0.0124787	0.0191079	0.0261865	0.0338574 (0.033771)
9	0.0	0.00700223	0.0142901	0.0221041	0.0306690 (0.030552)	0.0402273 (0.040113)
8	0.0	0.00798267	0.0165003	0.0259371 (0.025791)	0.0366711 (0.036505)	0.0491584 (0.049022)
7	0.0	0.00913658	0.0193053 (0.019149)	0.0311177 (0.030902)	0.0452843 (0.045060)	0.0628115 (0.062725)
6	0.0	0.0105684 (0.01045)	0.0231194 (0.022886)	0.0387258 (0.038410)	0.0589994 (0.058754)	0.0867292 (0.087229)
5	0.0	0.0125356 (0.012368)	0.0288587 (0.028507)	0.0513687 (0.050922)	0.0848470 (0.085152)	0.140496 (0.14898)
4	0.0	0.0155991 (0.015349)	0.0388162 (0.038242)	0.0771409 (0.076911)	0.153109 (0.17028)	
3	0.0	0.0211410 (0.020706)	0.0607198 (0.059794)	0.160034 (0.19684)		
2	0.0	0.0340844 (0.033085)	0.148554 (0.22997)			
1	0.0	0.0972028 (0.27096)				

and

$$q(\lambda, \mu, t) = \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} q_{mn}(t) S_{mn}(\varepsilon; \lambda, \mu). \quad (3.10)$$

An identical transform pair exists for  $\psi_{mn}(t)$  and  $\psi(\lambda, \mu, t)$ . Note that (3.9) can be obtained by multiplying (3.10) by  $S_{m'n'}^*(\lambda, \mu; \varepsilon)$ , integrating over the sphere, and using the orthonormality relation (3.7).

We now note the usefulness of spheroidal harmonic transforms in the solution of the invertibility principle (2.13). Substituting (3.10) and its companion for  $\psi$  into (2.13), and then

using (3.1), we immediately obtain

$$\psi_{mn} = -\frac{a^2 q_{mn}}{\alpha_{mn}(\varepsilon)}, \quad (3.11)$$

which provides the spectral space algebraic inversion of the potential vorticity  $q_{mn}$  to the streamfunction  $\psi_{mn}$ . The simple, convenient form of (3.11) depends on the formulation of the invertibility principle (2.13) as a relation between  $\psi$  and  $q$  rather than a relation between  $h$  and  $q$ . Since  $gh = 2\Omega\mu\psi$ , the relation between  $h$  and  $q$  is more complicated and not so easily solved by transform methods. Thus, in the formulation

**Table 4.** As in Table 2, but for  $\varepsilon = 1,000$ .

Eigenvalues:  $\alpha_{mn}(1,000)$

$n$	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
10	602.183	552.201	503.546	455.816	408.737	362.123
9	550.643	499.180	448.973	399.674	351.039	302.896
8	497.723	444.859	393.187	342.403	292.289	242.688
7	443.487	389.289	336.225	284.029	232.506	181.507
6	387.989	332.515	278.124	224.580	171.706	119.367
5	331.278	274.578	218.915	164.079	109.908	56.2790
4	273.396	215.515	158.629	102.550	47.1312	
3	214.381	155.358	97.2930	40.0158		
2	154.268	94.1385	34.9330			
1	93.0869	31.8832				
0	30.8666					

Dimensionless Frequencies:  $v_{mn}(1,000)/(2\Omega)$

$n$	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
10	0.0	0.00181094	0.00397183	0.00658161	0.00978623	0.0138075 (0.013779)
9	0.0	0.00200328	0.00445461	0.00750612	0.0113947 (0.011367)	0.0165073 (0.016472)
8	0.0	0.00224790	0.00508664	0.00876161 (0.0087377)	0.0136851 (0.013649)	0.0206026 (0.020563)
7	0.0	0.00256879	0.00594839 (0.0059307)	0.0105623 (0.010529)	0.0172039 (0.017156)	0.0275471 (0.027532)
6	0.0	0.00300738 (0.00299800)	0.00719104 (0.0071660)	0.0133583 (0.013311)	0.0232957 (0.023246)	0.0418877 (0.042244)
5	0.0	0.00364195 (0.0036284)	0.00913596 (0.0090976)	0.0182839 (0.018215)	0.0363940 (0.036556)	0.0888430 (0.11507)
4	0.0	0.00464005 (0.0046186)	0.0126080 (0.012542)	0.0292540 (0.029237)	0.0848694 (0.12515)	
3	0.0	0.00643674 (0.0063967)	0.0205565 (0.020441)	0.0749705 (0.13645)		
2	0.0	0.0106226 (0.010523)	0.0572525 (0.14907)			
1	0.0	0.0313645 (0.16309)				

of quasi-geostrophic theory on the sphere, it is preferable to express the invertibility principle as a relation between  $\psi$  and  $q$ , and to partition the flow into nondivergent and irrotational components rather than geostrophic and ageostrophic components. This avoids the computation of geostrophic velocities at and near the equator.

#### 4. Quasi-geostrophic Rossby-Haurwitz waves

Linearized Rossby-Haurwitz waves on a resting basic state are governed by (2.13) and the linearized version of (2.12). A rest-

ing basic state was chosen for simplicity and ease of comparison with shallow water primitive equation results, but it should be noted that the results shown here can also easily be extended to the case of a basic zonal flow with constant angular velocity. As is easily confirmed through the use of (3.1), the solutions for given values of  $m$ ,  $n$ , and  $\varepsilon$  are

$$q(\lambda, \mu, t) = \left( \frac{2E\alpha_{mn}(\varepsilon)}{a^2} \right)^{1/2} S_{mn}(\varepsilon; \mu) e^{i[m\lambda + v_{mn}(\varepsilon)t]} \quad (4.1)$$

**Table 5.** As in Table 2, but for  $\varepsilon = 10,000$ .

Eigenvalues:  $\alpha_{mn}(10,000)$

$n$	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
10	2042.66	1854.18	1666.80	1480.43	1295.02	1110.51
9	1853.07	1663.50	1475.01	1287.52	1100.97	915.323
8	1662.41	1471.75	1282.16	1093.55	905.889	719.107
7	1470.67	1278.94	1088.25	898.551	709.774	521.871
6	1277.87	1085.07	893.309	702.515	512.637	323.625
5	1084.01	890.164	697.330	505.455	314.487	124.375
4	889.116	694.219	500.325	307.379	115.329	
3	693.183	497.247	302.302	108.294		
2	496.221	299.256	103.268			
1	298.240	100.253				
0	99.2481					

Dimensionless Frequencies:  $v_{mn}(10,000)/(2\Omega)$

$n$	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
10	0.0	0.000539321	0.00119991	0.00202644	0.00308875	0.00450242 (0.0044992)
9	0.0	0.000601141	0.00135593	0.00233007	0.00363315 (0.0036303)	0.00546255 (0.0054587)
8	0.0	0.000679462	0.00155987	0.00274335 (0.0027410)	0.00441555 (0.0044116)	0.00695307 (0.0069485)
7	0.0	0.000781898	0.00183781 (0.0018361)	0.00333871 (0.0033355)	0.00563560 (0.0056305)	0.00958091 (0.0095802)
6	0.0	0.000921596 (0.00092084)	0.00223887 (0.0022365)	0.00427037 (0.0042656)	0.00780279 (0.0077971)	0.0154500 (0.015521)
5	0.0	0.00112339 (0.0011221)	0.00286808 (0.0028644)	0.00593524 (0.0059277)	0.0127191 (0.012746)	0.0402010 (0.078057)
4	0.0	0.00144047 (0.0014385)	0.00399740 (0.0039906)	0.00975994 (0.0097593)	0.0346833 (0.081982)	
3	0.0	0.00201107 (0.0020071)	0.00661590 (0.0066038)	0.0277024 (0.086147)		
2	0.0	0.00334162 (0.0033317)	0.0193670 (0.090556)			
1	0.0	0.00997475 (0.095215)				

and

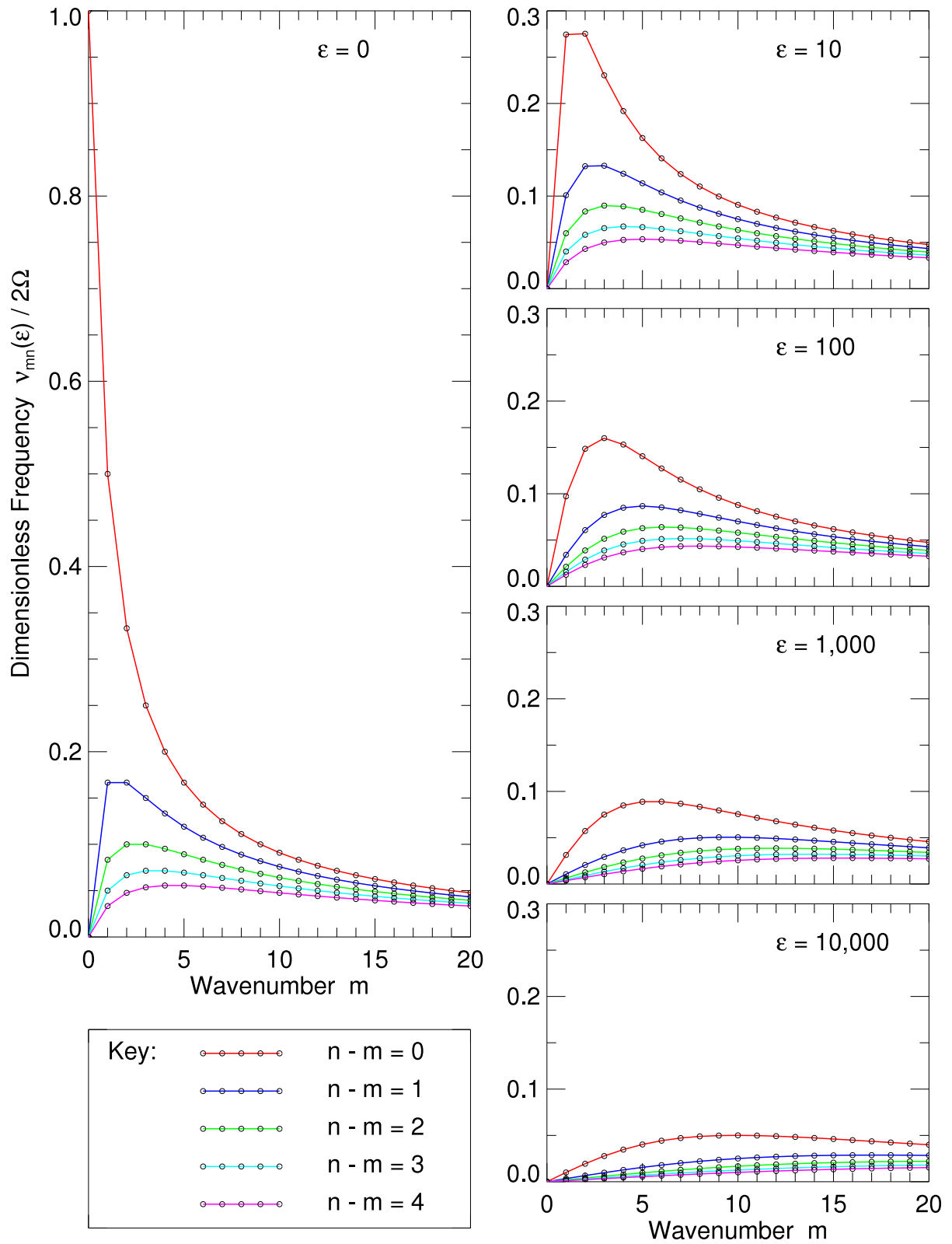
$$\psi(\lambda, \mu, t) = - \left( \frac{2a^2 E}{\alpha_{mn}(\varepsilon)} \right)^{1/2} S_{mn}(\varepsilon; \mu) e^{i[m\lambda + v_{mn}(\varepsilon)t]} \quad (4.2)$$

where  $E$  is a normalization constant and

$$v_{mn}(\varepsilon) = \frac{2\Omega m}{\alpha_{mn}(\varepsilon)} \quad (4.3)$$

is the Rossby-Haurwitz wave frequency. Note that in this section  $q$  and  $\psi$  are now the physical space fields corresponding

to a single spheroidal harmonic mode. Also note that the special case  $\varepsilon = 0$ , for which  $\alpha_{mn}(0) = n(n+1)$ , leads to the classical nondivergent barotropic model result  $\psi(\lambda, \mu, t) \sim P_n^m(\mu) \exp\{i[m\lambda + v_{mn}(0)t]\}$  with  $v_{mn}(0) = 2\Omega m/[n(n+1)]$ . Although the monochromatic wave given by (4.1) and (4.2) has been derived as a solution of the linearized PV equation, it is also a solution of the nonlinear PV equation (2.12). To see this, note that since the solution (4.1) for the  $q$  field and the solution (4.2) for the  $\psi$  field differ only by a constant factor, the isolines of  $\psi$  are always parallel to the isolines of  $q$ , which means that the Jacobian term in (2.12) vanishes. In



**Figure 1.** Dimensionless Rossby-Haurwitz wave frequencies,  $v_{mn}(\epsilon)/2\Omega$ , as a function of zonal wavenumber  $m$  for  $\epsilon = 0, 10, 100, 1000, 10000$ .



other words, (4.1) and (4.2) satisfy both the linear dynamical equations and the nonlinear dynamical equations. This result is a generalization of the results obtained by Craig (1945) and Neamtan (1946) for the nondivergent barotropic model (i.e., the  $\varepsilon = 0$  case).

Using the eigenvalues from the top half of Tables 2–5 in the Rossby-Haurwitz frequency formula (4.3), we obtain the dimensionless frequencies displayed in the bottom half of these tables and plotted in Fig. 1. Also displayed in parentheses in the bottom half of Tables 2–5 are the Rossby-Haurwitz wave frequencies computed by Longuet-Higgins (1968) for the shallow water primitive equations. These exact values are taken from Table 5 of Longuet-Higgins' paper. A careful inspection of Tables 2–5 shows that the quasi-geostrophic Rossby-Haurwitz wave frequencies (4.3) are very good approximations to the primitive equation frequencies, with the exception of low zonal wavenumber sectoral harmonics, i.e.,

small  $|m|$  for  $n - |m| = 0$ . For example, in the case  $\varepsilon = 10$ , the largest errors are for  $n = m = 1$  (34% error) and  $n = m = 2$  (9.9% error). One reason for these larger errors in the low zonal wavenumber sectoral harmonics is that the  $h$  and  $\psi$  fields vary as slowly as  $2\Omega\mu$ , so that the assumptions involved in approximating the linear balance relation (2.9) by the local linear balance relation (2.11) begin to break down. A second reason is that, in the primitive equation model, the low zonal wavenumber sectoral harmonics involve a combination of gravity wave dynamics and potential vorticity dynamics, and the quasi-geostrophic model is only able to accurately capture the potential vorticity part of the dynamics.

The streamfunction solution (4.2) contains all the information on the rotational part of the flow and on the deviations of the fluid depth. Thus, using (2.7) and (2.11) for given values of  $m$ ,  $n$ , and  $\varepsilon$ , the Rossby-Haurwitz wave solution can also be written as

$$\begin{pmatrix} u_\psi(\lambda, \mu, t) \\ v_\psi(\lambda, \mu, t) \\ h(\lambda, \mu, t) \end{pmatrix} = \left( \frac{2E}{\alpha_{mn}(\varepsilon)} \right)^{1/2} \begin{pmatrix} -(1-\mu^2)^{1/2} S'_{mn}(\varepsilon; \mu) \\ im(1-\mu^2)^{-1/2} S_{mn}(\varepsilon; \mu) \\ ag^{-1}2\Omega\mu S_{mn}(\varepsilon; \mu) \end{pmatrix} e^{i[m\lambda + v_{mn}(\varepsilon)t]} \quad (4.4)$$

where  $S'_{mn} = dS_{mn}/d\mu$ . From (4.4) we can show that every Rossby-Haurwitz wave eigenfunction has the same total energy, i.e.,

$$\begin{aligned} \frac{1}{8\pi} \int_{-1}^1 \int_0^{2\pi} \left( u_\psi u_\psi^* + v_\psi v_\psi^* + \frac{g}{h} h h^* \right) d\lambda d\mu \\ = \frac{E}{2\alpha_{mn}} \int_{-1}^1 \left[ (1-\mu^2) S_{mn}'^2 + \frac{m^2}{1-\mu^2} S_{mn}^2 + \varepsilon\mu^2 S_{mn}^2 \right] d\mu \\ = \frac{E}{2} \int_{-1}^1 S_{mn}^2 d\mu = E, \end{aligned} \quad (4.5)$$

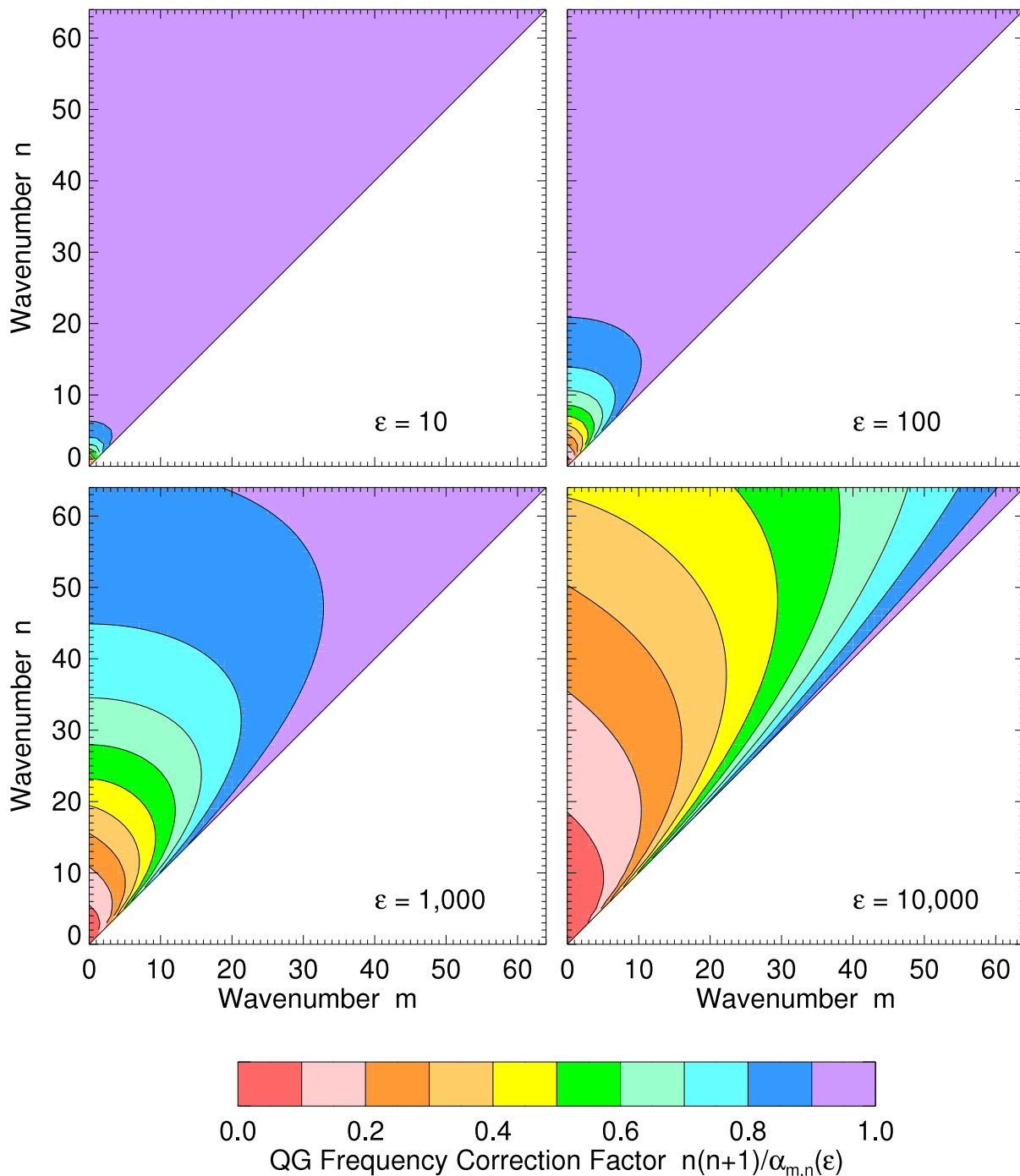
where the second equality follows from an integration by parts and the use of (3.3), and where the third equality follows from the orthonormality relation (3.8). The normalization constant  $E$  introduced in (4.1) and (4.2) can now be interpreted as the globally integrated total energy per unit mass for each Rossby-Haurwitz wave mode. Note that the condition  $n \gg \varepsilon^{1/2}$ , which is the condition that the spheroidal harmonics are well-approximated by the spherical harmonics, is also the condition that most of the energy is in kinetic form. With this energy normalization, each Rossby-Haurwitz wave eigenfunction has a different potential enstrophy. From (4.1) and (3.7) we can easily show that

$$\frac{1}{8\pi} \int_{-1}^1 \int_0^{2\pi} qq^* d\lambda d\mu = \frac{E\alpha_{mn}(\varepsilon)}{a^2}, \quad (4.6)$$

so that the potential enstrophy of a particular Rossby-Haurwitz wave eigenfunction is proportional to the spheroidal harmonic eigenvalue  $\alpha_{mn}(\varepsilon)$ , which, as shown in Tables 2–5, increases with  $n$  for fixed  $m$  and increases with  $|m|$  for fixed  $n - |m|$ .

In the very early days of numerical weather prediction, forecasts were made over a regional domain using the nondivergent barotropic model. As computer power increased, the barotropic model domain became nearly hemispheric. When these nearly hemispheric forecasts were compared to observations, it was soon noticed (Wolff 1958; Cressman 1958; Wiin-Nielsen 1959) that the nondivergent model produced an erroneous westward propagation of ultralong waves. As discussed by Phillips (2000), this error was initially reduced by "artificially holding those wavenumbers fixed during the integration. Later, Cressman (1958) obtained a slightly better error reduction by introducing an empirical correction for the divergence that can be present in a barotropic model with a free surface." Phillips also noted that "it is not easy to think of an improvement to the empirical corrections introduced by Cressman." This issue can now be reexamined by writing (4.3) in the form

$$v_{mn}(\varepsilon) = \left( \frac{n(n+1)}{\alpha_{mn}(\varepsilon)} \right) \left( \frac{2\Omega m}{n(n+1)} \right), \quad (4.7)$$



**Figure 2.** Isolines of the “quasi-geostrophic frequency correction factor,” defined as  $n(n+1)/\alpha_{mn}(\epsilon)$ , for  $\epsilon = 10, 100, 1000, 10000$ .

and noting that the factor  $n(n+1)/\alpha_{mn}(\epsilon)$  modifies the non-divergent Rossby-Haurwitz wave frequency  $2\Omega m/[n(n+1)]$  to the quasi-geostrophic Rossby-Haurwitz wave frequency  $\nu_{mn}(\epsilon)$ . The “quasi-geostrophic frequency correction factor”  $n(n+1)/\alpha_{mn}(\epsilon)$  represents a theoretical alternative to the empirical correction factor of Cressman. Isolines of this frequency correction factor in  $(m, n)$ -space for  $\epsilon = 10, 100, 1000, 10000$  are presented in the four panels of Fig. 2.

For  $\epsilon = 10$  the correction factor is near unity except for the ultralong waves  $m = 1, 2, 3$ . As  $\epsilon$  increases, the correction factor becomes significantly less than unity over a growing region of wavenumber space. Thus, the quasi-geostrophic Rossby-Haurwitz wave frequency  $\nu_{mn}(\epsilon)$  is consistently smaller than the corresponding barotropic frequency. Since the spheroidal harmonic eigenvalue  $\alpha_{mn}(\epsilon)$  arises from the operator  $\nabla^2 - \epsilon\mu^2/a^2$ , while the spherical harmonic eigenvalue  $n(n+1)$

arises from the operator  $\nabla^2$ , the factor  $n(n+1)/\alpha_{mn}(\varepsilon)$  arises because the quasi-geostrophic model makes a mathematical distinction between vorticity and potential vorticity while the barotropic model does not. We can summarize this discussion by noting that the whole problem of erroneous westward propagation of ultralong waves could have been avoided if early attempts at prediction using barotropic models had been based on the formulation (2.12)–(2.13) rather than the nondivergent barotropic formulation.

During the final review stage of this paper, we were made aware of recent work of Verkley (2009), who has independently reached many of the conclusions given here in Section 4. In particular, he has presented a detailed comparison of the eigenvalues and eigenfunctions of the quasi-geostrophic Rossby-Haurwitz waves with the corresponding primitive equation results of Longuet-Higgins (1968). He has also presented an interesting argument that legitimizes, to some extent, the common practice of approximating  $\mu^2$  in (2.13) by a constant. Verkley's results extend those of Wun-

derer (2001), who has compared the Rossby-Haurwitz eigenvalues for the linear balance model (see also Moura 1976) and the quasi-geostrophic model with those calculated from the hemispheric semigeostrophic theory of Magnusdottir and Schubert (1991).

## 5. Quasi-geostrophic turbulence on the sphere

The total energy principle, obtained by multiplying (2.12) by  $-\psi$  and integrating over the sphere, is  $dE/dt = 0$ , where

$$\begin{aligned} E &= \frac{1}{8\pi} \int_{-1}^1 \int_0^{2\pi} \left( \nabla\psi \cdot \nabla\psi + \frac{\varepsilon\mu^2}{a^2} \psi^2 \right) d\lambda d\mu \\ &= -\frac{1}{4\pi} \int_{-1}^1 \int_0^{2\pi} \frac{1}{2} \psi q d\lambda d\mu \end{aligned} \quad (5.1)$$

is the total energy. Using the spheroidal harmonic expansions for  $\psi(\lambda, \mu, t)$  and  $q(\lambda, \mu, t)$ , we can express  $E$  in spectral space as

$$\begin{aligned} E &= -\frac{1}{4\pi} \int_{-1}^1 \int_0^{2\pi} \frac{1}{2} \left( \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \psi_{mn} \mathcal{S}_{mn} \right) \left( \sum_{m'=-\infty}^{\infty} \sum_{n'=|m'|}^{\infty} q_{m'n'}^* \mathcal{S}_{m'n'}^* \right) d\lambda d\mu \\ &= -\sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \sum_{m'=-\infty}^{\infty} \sum_{n'=|m'|}^{\infty} \frac{1}{2} \psi_{mn} q_{m'n'}^* \left( \frac{1}{4\pi} \int_{-1}^1 \int_0^{2\pi} \mathcal{S}_{mn} \mathcal{S}_{m'n'}^* d\lambda d\mu \right) \\ &= -\sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \frac{1}{2} \psi_{mn} q_{mn}^* = \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \frac{\alpha_{mn}}{2a^2} |\psi_{mn}|^2 = \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} E_{mn}, \end{aligned} \quad (5.2)$$

where we have used the orthonormality relation (3.7) to obtain the first equality in the last line, the invertibility principle  $q_{mn}^* = -a^{-2} \alpha_{mn} \psi_{mn}^*$  to obtain the second equality in the last line, and where we have defined

$$E_{mn} = \frac{\alpha_{mn}}{2a^2} |\psi_{mn}|^2. \quad (5.3)$$

Equation (5.2) is the ‘Parseval relation’ for the total energy. It allows us to compute the total energy of the atmosphere either by an integral of  $\frac{1}{2}(\nabla\psi \cdot \nabla\psi + \varepsilon a^{-2} \mu^2 \psi^2)$  over physi-

cal space or by a sum of  $\frac{1}{2} a^{-2} \alpha_{mn} |\psi_{mn}|^2$  over spheroidal harmonic wavenumber space.

The potential enstrophy principle, obtained by multiplying (2.12) by  $q$  and integrating over the sphere, is  $dZ/dt = 0$ , where

$$Z = \frac{1}{4\pi} \int_{-1}^1 \int_0^{2\pi} \frac{1}{2} q^2 d\lambda d\mu \quad (5.4)$$

is the potential enstrophy. Using the spheroidal harmonic expansion for  $q(\lambda, \mu, t)$ , we can express  $Z$  in spectral space as

$$\begin{aligned} Z &= \frac{1}{4\pi} \int_{-1}^1 \int_0^{2\pi} \frac{1}{2} \left( \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} q_{mn} \mathcal{S}_{mn} \right) \left( \sum_{m'=-\infty}^{\infty} \sum_{n'=|m'|}^{\infty} q_{m'n'}^* \mathcal{S}_{m'n'}^* \right) d\lambda d\mu \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \sum_{m'=-\infty}^{\infty} \sum_{n'=|m'|}^{\infty} \frac{1}{2} q_{mn} q_{m'n'}^* \left( \frac{1}{4\pi} \int_{-1}^1 \int_0^{2\pi} \mathcal{S}_{mn} \mathcal{S}_{m'n'}^* d\lambda d\mu \right) \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} \frac{1}{2} q_{mn} q_{mn}^* = \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} Z_{mn}, \end{aligned} \quad (5.5)$$

where we have again used the orthonormality relation (3.7) to obtain the last line in (5.5). Using (5.3), this can also be written as

$$Z_{mn} = \frac{1}{2} |q_{mn}|^2 = \frac{\alpha_{mn}}{a^2} E_{mn}. \quad (5.6)$$

Equation (5.5) is the ‘Parseval relation’ for the potential enstrophy. It allows us to compute the total potential enstrophy either by an integral of  $\frac{1}{2}q^2$  over physical space or by a sum of  $\frac{1}{2}|q_{mn}|^2$  over spheroidal harmonic wavenumber space.

As discussed by Charney (1971), energy moves toward lower wavenumber while potential enstrophy moves toward higher wavenumber. Because of this cascade in opposite directions, potential enstrophy is more subject to small scale dissipation, and hence can be selectively decayed while energy is nearly conserved even in the presence of dissipation. However, as energy moves to lower wavenumber, the effects of the earth’s sphericity become important, and large-scale structures begin to elongate in the east-west direction. This ‘Rhines barrier’ has traditionally been thought of in a one-dimensional wavenumber (or total wavenumber) space. For barotropic flows, the concept has been extended to two-dimensional wavenumber space by Vallis and Maltrud (1993) and Huang and Robinson (1998). As the energy cascades towards smaller wavenumbers, an ‘anisotropic Rhines barrier’ is reached, so that the cascade tends to proceed farther in the region of wavenumber space where the zonal wavenumber is small. The spherical quasi-geostrophic theory presented here provides a framework for extending these concepts from two-dimensional wavenumber space to three-dimensional wavenumber space. Instead of viewing the ‘anisotropic Rhines barrier’ as a line in two-dimensional wavenumber space  $(m, n)$ , we can view the barrier as a surface in three-dimensional wavenumber space  $(m, n, \varepsilon)$ .

To quantify these ideas, we first note that (2.12) contains nonlinear advection of  $q$  and a linear term associated with Rossby-Haurwitz waves. The Rossby-Haurwitz wave frequency is given by  $2\Omega m/\alpha_{mn}(\varepsilon)$ . The turbulent frequency is given by  $a^{-1}[\alpha_{mn}(\varepsilon)]^{1/2} V_{\text{rms}}$ , where  $V_{\text{rms}}$  is the root-mean-square velocity. The dynamics is wavelike if

$$2\Omega m/\alpha_{mn}(\varepsilon) \gg a^{-1}[\alpha_{mn}(\varepsilon)]^{1/2} V_{\text{rms}},$$

while it is dominated by turbulence if

$$2\Omega m/\alpha_{mn}(\varepsilon) \ll a^{-1}[\alpha_{mn}(\varepsilon)]^{1/2} V_{\text{rms}}.$$

The anisotropic Rhines barrier is defined by equating the two time scales, which, after some rearrangement, leads to

$$\frac{m}{[\alpha_{mn}(\varepsilon)]^{3/2}} = \frac{V_{\text{rms}}}{2\Omega a}. \quad (5.7)$$

For a given  $V_{\text{rms}}/(2\Omega a)$  and a given  $\varepsilon$ , (5.7) defines a curve in the spheroidal harmonic wavenumber plane  $(m, n)$ . Several such curves, for  $2\Omega a = 929 \text{ ms}^{-1}$  and for different values of  $V_{\text{rms}}$ , are displayed in Fig. 3. For a given  $V_{\text{rms}}$ , the region below the appropriate Rhines curve is wavelike, while the region

above the curve is dominated by turbulence. By superposing these figures one can visualize a three-dimensional Rhines surface that dictates the lowest wavenumbers to which a turbulent flow can cascade.

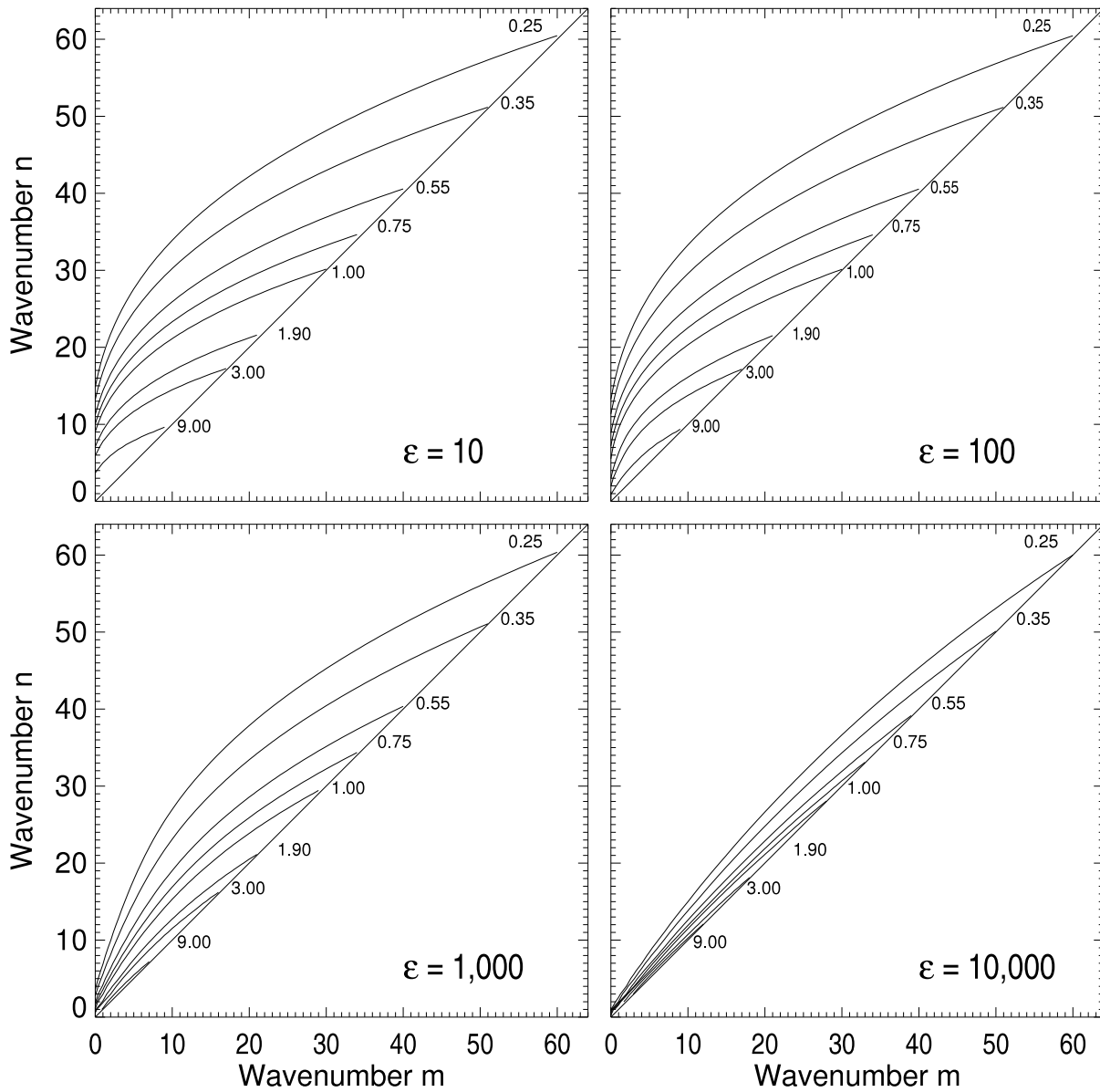
## 6. Concluding remarks

In the historical development of quasi-geostrophic theory (see Phillips 1963 for a review) the midlatitude  $\beta$ -plane version has played the lead role. However, in the modern era of global climate models, a fully spherical version of quasi-geostrophic theory is useful for a variety of purposes. We have discussed a form of shallow water quasi-geostrophic theory that can describe low Rossby number flows on the entire sphere—a form that dates back to the pioneering work of Kuo (1959) and Charney and Stern (1962). The theory is concisely stated by the potential vorticity conservation relation (2.12) and the invertibility principle (2.13). Except for sectoral modes of low zonal wavenumber, this theory accurately approximates the Rossby-Haurwitz wave spectrum associated with the primitive equations. This makes the theory (or its extension to the continuously stratified case) useful for application to such problems as Rossby-Haurwitz wave dispersion on the sphere and the vertical propagation of Rossby-Haurwitz waves into the stratosphere.

The global quasi-geostrophic theory presented here has incorporated several important ingredients:

- (i) To obtain an invertibility principle from (2.8) we have chosen to leave  $\nabla^2\psi$  unchanged and to approximate  $gh$  by  $2\Omega\mu\psi$ , rather than leaving  $gh$  unchanged and approximating  $\nabla^2\psi$  in terms of  $gh$ .
- (ii) The flow partitioning is between nondivergent and irrotational components rather than between geostrophic and ageostrophic components.
- (iii) Spheroidal harmonics are introduced and utilized to transform the theory into spectral space, an extension that simplifies the invertibility principle and facilitates the understanding of Rossby-Haurwitz waves and the cascades of energy and potential enstrophy.

To the authors’ knowledge, all three of these ingredients have not been fully brought together before to exploit the advantages of global quasi-geostrophic theory, particularly ingredient (iii). Ingredients (i) and (ii) are not new and can be found together in previous literature, such as the classic papers of Kuo (1959) and Charney and Stern (1962) and also the more recent work of Verkley (2009). Although the works of Kuo and Verkley do recognize a connection of this theory with spheroidal harmonics, neither of these works fully develop and utilize this important connection. We also note that spheroidal harmonics are briefly discussed by Longuet-Higgins (1965, 1968) in the context of asymptotic approximations to Laplace’s tidal equations and by Dickinson (1968) in the context of the vertical propagation of planetary waves from the troposphere into the stratosphere.



**Figure 3.** Anisotropic Rhines curves in the wavenumber plane of spheroidal harmonics  $S_{mn}(\epsilon; \lambda, \mu) = S_{mn}(\epsilon; \mu)e^{im\lambda}$  for select  $\epsilon$ , where  $m$  is the zonal wavenumber and  $n$  is the total wavenumber. The curves are based on (5.7), with the values of  $V_{rms}$  labeled in  $m\ s^{-1}$  to the right of the diagonal.

It is interesting to note that Matsuno (1970) has formulated a linear version of global quasi-geostrophic theory that does not use ingredients (i) or (ii). In that work, the flow partition is into geostrophic and ageostrophic components, and the invertibility principle is expressed in terms of the geopotential rather than the streamfunction. To the authors' knowledge, a full nonlinear version of Matsuno's model has not been developed. Finally, we refer to the work of Karoly and Hoskins (1982), Mak (1991), Marshall and Molteni (1993), and Theiss (2004), who studied the equatorward energy cascade, critical latitude, and the predominance of cyclonic vortices in geostrophic turbulence. The theories presented in these papers are global to a certain extent, but do not include the full

variability of the Coriolis parameter since their invertibility principles use a constant Rossby length rather than the  $\epsilon\mu^2/a^2$  form of (2.13).

Some of the issues that have arisen here have also played important roles in the history of numerical weather prediction. Not long after the encouraging results obtained by Charney et al. (1950) using the barotropic model, Charney (1954) constructed a 3-level quasi-geostrophic model for the numerical prediction of cyclogenesis. As shown by Phillips (1958, 2000), this model failed to produce an accurate forecast of storm location for an interesting example of cyclogenesis on the east coast of the United States. In his analysis of the cause of this failure, Phillips concluded that the problem was in the

use of geopotential rather than streamfunction. For example, in the shallow water context, the formula for the potential vorticity anomaly,  $q = \nabla^2\psi - (2\Omega\mu/\bar{h})h$ , and the conservation law for  $q$  can be approximated in two ways:

- (A) Leave the  $(2\Omega\mu/\bar{h})h$  term in its present form, approximate the vorticity  $\nabla^2\psi$  by its geostrophic value (i.e., in terms of  $\nabla^2h$ ), and advect  $q$  using the geostrophic wind.
- (B) Leave the  $\nabla^2\psi$  term in its present form, approximate the  $(2\Omega\mu/\bar{h})h$  term by  $(\varepsilon\mu^2/a^2)\psi$ , and advect  $q$  using the nondivergent wind  $(u_\psi, v_\psi)$ .

Method (A), the conventional method used by Charney (1954), treats the initial mass field as “exact”, while the accuracy of the initial wind field is limited by the geostrophic approximation. In contrast, method (B) treats the initial nondivergent wind as “exact”, while the accuracy of the initial mass field is limited by the local linear balance approximation  $gh = 2\Omega\mu\psi$ . Phillips pointed out that ordinary scale analysis is not useful in deciding which method is best. However, he argues that experience shows that method (B) seems to be best because typical atmospheric flow patterns are such that it is more important to have a good representation of the wind rather than the temperature. In fact, the failure of Charney’s (1954) model to accurately forecast storm location during cyclogenesis can be attributed to the erroneously large PV advection by the geostrophic winds. Phillips concludes that “quasi-geostrophic models might have been more productive in the United States if they had used streamfunction in place of geopotential.” Unfortunately, “this possibility for improvement in the quasi-geostrophic model was realized too late,” since interest was shifting to the use of the primitive equations. Concerning the relative merits of methods (A) and (B), we have reached the same conclusion as Phillips, but our arguments deal with the extension of quasi-geostrophic theory to the entire sphere rather than the forecast accuracy of mid-latitude cyclogenesis. Thus, there are at least two good reasons to prefer method (B) over method (A).

In closing we note that a wide variety of finite difference and spectral methods can be used to solve (2.12) and (2.13). In terms of spectral methods we have the choice of using either spheroidal harmonics or spherical harmonics as basis functions. The advantage of the former is that the spheroidal harmonic transform of (2.13) is a simple (i.e., diagonal) algebraic relation between the spectral coefficients of  $\psi$  and  $q$ , as given by (3.11). In contrast, the spherical harmonic transform of (2.13) results in a tridiagonal algebraic system relating the spectral coefficients of  $\psi$  and  $q$ . Thus, the use of spheroidal harmonic basis functions may be preferable in the shallow water context. However, in a multilayer model of a continuously stratified fluid we obtain an invertibility principle of the form (2.13) for each vertical mode, with each mode having a distinct value of  $\varepsilon$ . Then, for efficient numerical integrations, we would need to precompute and store spheroidal harmonic basis sets for each  $\varepsilon$ . Thus, for multilayer models, it is probably preferable to use a spherical harmonic basis set,

even though the resulting algebraic form of (2.13) is tridiagonal rather than diagonal. However, it should be noted that, even though an efficient numerical integration might use a spherical harmonic basis, the diagnostic analysis of energy and potential enstrophy cascades could be more insightful in the context of spheroidal harmonic basis functions. Further discussion and applications of the continuously stratified version of quasi-geostrophic theory on the sphere are given in Silvers (2007). A detailed analysis of the appropriate spectral methods for the stratified case will be presented in a forthcoming paper.

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## Appendix A: Calculation of the eigenvalues and eigenfunctions of the spheroidal wave equation

The eigenvalues  $\alpha_{mn}(\varepsilon)$  and the eigenfunctions  $S_{mn}(\varepsilon; \mu)$  of the spheroidal wave equation (3.3) are computed using a method adapted from Hodge (1970). The key step of this method is to consider the associated Legendre expansion

$$S_{mn}(\varepsilon; \mu) = \sum_{r=0,1}^{\infty} d_r^{mn}(\varepsilon) P_{m+r}^m(\mu), \quad (\text{A.1})$$

where the summation begins at  $r = 0$  and runs over even integers if  $n - |m|$  is even, or begins at  $r = 1$  and runs over odd integers if  $n - |m|$  is odd, and where  $d_r^{mn}(\varepsilon)$  are the expansion coefficients of  $S_{mn}(\varepsilon; \mu)$  in terms of the associated Legendre functions  $P_n^m(\mu)$ . The associated Legendre functions satisfy the differential equation

$$\frac{d}{d\mu} \left[ (1 - \mu^2) \frac{dP_n^m}{d\mu} \right] + \left[ n(n+1) - \frac{m^2}{1 - \mu^2} \right] P_n^m = 0. \quad (\text{A.2})$$

and the orthonormality condition

$$\begin{aligned} & \frac{1}{2} \int_{-1}^1 P_n^m(\mu) P_{n'}^m(\mu) d\mu \\ &= \begin{cases} \frac{1}{(2n+1)} \frac{(n+m)!}{(n-m)!} & n' = n \\ 0 & n' \neq n. \end{cases} \end{aligned} \quad (\text{A.3})$$

Note that  $S_{mn}(\varepsilon; \mu)$  and  $P_n^m(\mu)$  are both even functions about  $\mu = 0$  if  $n - |m|$  is an even integer, while they are both odd functions about  $\mu = 0$  if  $n - |m|$  is an odd integer. Substituting the associated Legendre expansion (A.1) into the ordinary

differential equation (3.3), and then using (A.2), we obtain

$$\sum_{r=0,1}^{\infty} d_r^{mn}(\varepsilon) \left[ \alpha_{mn}(\varepsilon) - (m+r)(m+r+1) - \varepsilon \mu^2 \right] P_{m+r}^m(\mu) = 0. \quad (\text{A.4})$$

Since the associated Legendre functions satisfy the recurrence relation

$$P_{n+1}^m(\mu) = \frac{(2n+1)\mu P_n^m(\mu) - (n+m)P_{n-1}^m(\mu)}{(n-m+1)}, \quad (\text{A.5})$$

or equivalently

$$\mu P_n^m(\mu) = \frac{(n+m)P_{n-1}^m(\mu) + (n-m+1)P_{n+1}^m(\mu)}{(2n+1)},$$

we can easily show that

$$\begin{aligned} \mu^2 P_n^m(\mu) &= \frac{(n-m+1)(n-m+2)}{(2n+1)(2n+3)} P_{n+2}^m(\mu) \\ &+ \frac{2n(n+1) - 2m^2 - 1}{(2n-1)(2n+3)} P_n^m(\mu) \\ &+ \frac{(n+m-1)(n+m)}{(2n-1)(2n+1)} P_{n-2}^m(\mu). \end{aligned} \quad (\text{A.6})$$

When (A.6) is used in the last term on the left hand side of (A.4), we immediately conclude that the expansion coefficients  $d_r^{mn}(\varepsilon)$  must satisfy the three term recurrence relation

$$A_r^m(\varepsilon) d_{r+2}^{mn}(\varepsilon) + [B_r^m(\varepsilon) - \alpha_{mn}(\varepsilon)] d_r^{mn}(\varepsilon) + C_r^m(\varepsilon) d_{r-2}^{mn}(\varepsilon) = 0 \quad \text{for } r \geq 0, \quad (\text{A.7})$$

where

$$A_r^m(\varepsilon) = \left( \frac{(2m+r+2)(2m+r+1)}{(2m+2r+3)(2m+2r+5)} \right) \varepsilon, \quad (\text{A.8})$$

$$B_r^m(\varepsilon) = (m+r)(m+r+1) + \left( \frac{2(m+r)(m+r+1) - 2m^2 - 1}{(2m+2r-1)(2m+2r+3)} \right) \varepsilon, \quad (\text{A.9})$$

$$C_r^m(\varepsilon) = \left( \frac{r(r-1)}{(2m+2r-3)(2m+2r-1)} \right) \varepsilon. \quad (\text{A.10})$$

When the system (A.7) for the expansion coefficients  $d_r^{mn}(\varepsilon)$  is written in matrix form, the nonzero elements of the matrix are seen to be on the main diagonal and a distance two above and below the main diagonal. To convert the problem (A.7) to a simple tridiagonal form we introduce the reindexing

$$\begin{aligned} D_q &= C_{2q+s}^m(\varepsilon), & E_q &= B_{2q+s}^m(\varepsilon), \\ F_q &= A_{2q+s}^m(\varepsilon), & a_q &= d_{2q+s}^{mn}(\varepsilon), \end{aligned} \quad (\text{A.11})$$

where  $s = 0$  if  $n - |m|$  is even and  $s = 1$  if  $n - |m|$  is odd. For simplicity we have omitted the relevant superscripts from  $D_q$ ,  $E_q$ ,  $F_q$ , and  $a_q$ . Using these definitions, (A.7) takes the tridiagonal form

$$D_q a_{q-1} + (E_q - \alpha) a_q + F_q a_{q+1} = 0 \quad \text{for } q \geq 0. \quad (\text{A.12})$$

We can now convert (A.12) to a tridiagonal, symmetric matrix problem by considering the new variable  $b_q$ , defined in terms of  $a_q$  by

$$a_q = \left( \frac{D_1 D_2 D_3 \cdots D_q}{F_0 F_1 F_2 \cdots F_{q-1}} \right)^{1/2} b_q. \quad (\text{A.13})$$

Substituting (A.13) into (A.12) and multiplying the result by  $[(F_0 F_1 F_2 \cdots F_{q-1}) / (D_1 D_2 D_3 \cdots D_q)]^{1/2}$ , we obtain

$$\begin{aligned} (D_q F_{q-1})^{1/2} b_{q-1} + (E_q - \alpha) b_q \\ + (D_{q+1} F_q)^{1/2} b_{q+1} = 0 \end{aligned} \quad (\text{A.14})$$

for  $q \geq 0$ , which can also be written in the matrix form

$$\begin{pmatrix} E_0 & (D_1 F_0)^{1/2} & 0 & 0 & \cdots \\ (D_1 F_0)^{1/2} & E_1 & (D_2 F_1)^{1/2} & 0 & \cdots \\ 0 & (D_2 F_1)^{1/2} & E_2 & (D_3 F_2)^{1/2} & \cdots \\ 0 & 0 & (D_3 F_2)^{1/2} & E_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} = \alpha \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}. \quad (\text{A.15})$$

To summarize, the eigenvalues  $\alpha_{mn}(\varepsilon)$  and eigenfunctions  $S_{mn}(\varepsilon; \mu)$  for a given  $\varepsilon$  are computed as follows. For a particular zonal wavenumber  $m$ , first choose  $s = 0$  and then compute  $D_q, E_q, F_q$  from (A.8)–(A.11). After solving the symmetric, tridiagonal eigensystem problem (A.15), convert  $b_q$  to  $a_q$  via (A.13), and then  $a_q$  to  $d_{2q+s}^{mn}$  via the last entry in (A.11). This whole procedure is then repeated for  $s = 1$ , the net result being that we have found all the eigenvalues  $\alpha_{mn}(\varepsilon)$  and expansion coefficients  $d_r^{mn}(\varepsilon)$  for a given zonal wavenumber  $m$ . Since the eigenfunctions  $S_{mn}(\varepsilon; \mu)$  are only determined to within a multiplicative constant, the final step is to normalize them in such a way that (3.8) is satisfied.

In practice the number of terms in the summation on the right hand side of (A.1) and the number of rows and columns in (A.15) must be finite. Since our global quasi-geostrophic model uses triangular truncation with maximum wavenumber  $M$ , the sums in (3.10) are over  $|m| \leq M$  and  $n \leq M$ , and we must accurately compute the eigenvalues  $\alpha_{mn}(\varepsilon)$  and the eigenfunctions  $S_{mn}(\varepsilon; \mu)$  over this range of  $m$  and  $n$ . Thus, the upper limit of the sum in (A.1) must be larger than  $M$ . However, it need only be slightly larger than  $M$  because of the rapid convergence of the associated Legendre expansion (A.1).

## Appendix B: Asymptotic forms of the eigenvalues and eigenfunctions for large epsilon

To derive asymptotic forms for the eigenvalues  $\alpha_{mn}(\varepsilon)$  and eigenfunctions  $S_{mn}(\mu; \varepsilon)$  we first note that the substitution  $S_{mn}(\mu; \varepsilon) = (1 - \mu^2)^{m/2} U_{mn}(\mu; \varepsilon)$  transforms the ordinary differential equation (3.3) to

$$(1 - \mu^2) \frac{d^2 U_{mn}}{d\mu^2} - 2(m+1)\mu \frac{dU_{mn}}{d\mu} + [\alpha_{mn} - m(m+1) - \varepsilon\mu^2] U_{mn} = 0. \quad (\text{B.1})$$

Defining  $y = \varepsilon^{1/4} \mu$ , (B.1) becomes

$$\left( \varepsilon^{1/2} - y^2 \right) \frac{d^2 U_{mn}}{dy^2} - 2(m+1)y \frac{dU_{mn}}{dy} + [\alpha_{mn} - m(m+1) - \varepsilon^{1/2} y^2] U_{mn} = 0. \quad (\text{B.2})$$

When  $\varepsilon \rightarrow \infty$  and  $y^2 \ll \varepsilon^{1/2}$ , (B.2) is approximated by

$$\frac{d^2 U_{mn}}{dy^2} + (\varepsilon^{-1/2} \alpha_{mn} - y^2) U_{mn} = 0, \quad (\text{B.3})$$

an equation that plays a central role in equatorial  $\beta$ -plane theory (Matsuno 1966). Bounded solutions  $\exp(-\frac{1}{2}y^2) H_r(y)$  exist when  $\varepsilon^{-1/2} \alpha_{mn} = 2r + 1$  for  $r = 0, 1, 2, \dots$ , where  $H_r(y)$  is the Hermite polynomial of order  $r$ . Thus, as  $\varepsilon \rightarrow \infty$ , the spheroidal function  $S_{mn}(\mu; \varepsilon)$  becomes proportional to  $\exp(-\frac{1}{2}y^2) H_r(y)$ . To determine  $r$  we argue as follows. For given  $m, n$ ,  $S_{mn}(\mu; \varepsilon)$  has the same number of zeroes no matter what the value of  $\varepsilon$ . Since  $S_{mn}(\mu; 0) = P_{mn}(\mu)$ , and since

the associated Legendre function  $P_{mn}(\mu)$  has  $n - m$  zeroes in the range  $-1 < \mu < 1$ ,  $S_{mn}(\mu; \varepsilon)$  also has  $n - m$  zeroes in this range. Since the Hermite polynomial  $H_r(y)$  has  $r$  zeroes in the range  $-\infty < y < \infty$ , we conclude that  $S_{mn}(\mu; \varepsilon)$  becomes proportional to  $\exp(-\frac{1}{2}y^2) H_{n-m}(y)$  as  $\varepsilon \rightarrow \infty$ . In summary,

$$S_{mn}(\mu; \varepsilon) \sim \exp\left(-\frac{1}{2}\varepsilon^{1/2}\mu^2\right) H_{n-m}(\varepsilon^{1/4}\mu) \quad (\text{B.4})$$

and

$$\alpha_{mn}(\varepsilon) \sim \varepsilon^{1/2} [1 + 2(n - m)] \quad (\text{B.5})$$

as  $\varepsilon \rightarrow \infty$ . It should be noted that, while the derivation of (B.5) shows the connection with equatorial  $\beta$ -plane theory, the asymptotic formula (3.6) is considerably more accurate than (B.5).

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